

Hamiltonian analysis of non-projectable modified $F(R)$ Hořava-Lifshitz gravity

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Abstract

We study a version of the recently proposed modified $F(R)$ Hořava-Lifshitz gravity that abandons the projectability condition of the lapse variable. We discovered that the projectable version of this theory has a consistent Hamiltonian structure, and that the theory has interesting cosmological solutions which can describe the eras of accelerated expansion of the universe in a unified manner. The usual Hořava-Lifshitz gravity is a special case of our theory. Hamiltonian analysis of the non-projectable theory, however, shows that this theory has serious problems. These problems are compared with those found in the original Hořava-Lifshitz gravity. A general observation on the structure of the Poisson bracket of Hamiltonian constraints in all theories of the Hořava-Lifshitz type is made: in the resulting tertiary constraint the highest order spatial derivative of the lapse N is always of uneven order. Since the vanishing of the lapse ($N = 0$) is required by the preservation of the Hamiltonian constraints under time evolution, we conclude that the non-projectable version of the theory is physically inconsistent.

1 Introduction

Last year the so-called Hořava-Lifshitz theory of gravity was proposed [1] (see also [2, 3]). This theory is a candidate for a quantum field theory of gravity that aims to provide an ultraviolet (UV) completion of General Relativity (GR). At short distances it describes interacting nonrelativistic gravitons. Hořava-Lifshitz gravity exhibits anisotropic scaling of space and time coordinates

$$\boldsymbol{x} \rightarrow b\boldsymbol{x}, \quad t \rightarrow b^z t \quad (1.1)$$

with a dynamic critical exponent $z = 1, 2, 3, \dots$. In the UV regime the value of the critical exponent z is chosen so that the gravitational coupling constant κ^2 is dimensionless. In $(D + 1)$ -dimensional space-time we have the scaling dimension $[\kappa^2] = z - D$. Thus the choice $z = D$ is argued to ensure that the theory is power-counting renormalizable. For the usual case of 3-dimensional space, $D = 3$, we choose $z = 3$.

The space-time manifold \mathcal{M} is assumed to possess a foliation structure that enables one to define \mathcal{M} as a union of space-like hypersurfaces Σ_t of constant time t . Due to the foliation the space-time is invariant under the foliation-preserving diffeomorphisms, whose infinitesimal generators are of the form

$$\delta\boldsymbol{x} = \boldsymbol{\zeta}(t, \boldsymbol{x}), \quad \delta t = f(t), \quad (1.2)$$

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instead of the full diffeomorphism invariance of GR. For simplicity the topological structure of space-time is assumed to be such that every leaf Σ_t of the foliation is equivalent to a fixed manifold Σ : $\mathcal{M} \cong \mathbb{R} \times \Sigma$. The preferred foliation of \mathcal{M} enables the inclusion of spatial covariant derivatives into the action, which improve the UV behaviour, while avoiding time derivatives higher than the second order, which are known to produce problematic ghosts.

At low energies and large distances the critical exponent is expected to flow to $z = 1$, so that the theory can coincide with GR. The Lorentz symmetry emerges at low energies as an accidental or approximate symmetry, but it is absent in the fundamental description.

The theory comes in two flavors, with or without the projectability condition that requires the lapse to depend only on the time coordinate, $N = N(t)$. The projectability condition is one of the features that makes the theory differ from GR. Note, however, that many solutions of GR, as well as of non-projectable Hořava-Lifshitz gravity, respect the condition $N = N(t)$ even though the general theory does not.

In the original theory an additional symmetry, the condition of detailed balance, is assumed. It defines the potential part of the gravitational action in terms of a variation of a D -dimensional action on the spatial hypersurface with respect to the spatial metric. The purpose of the detailed balance condition is to reduce the number of independent couplings to 3 — otherwise there are 9 independent couplings. We, however, do not assume this condition, so all terms that have appropriate scaling properties and that are covariant under foliation-preserving diffeomorphisms can be included.

This theory has received a lot of attention and many potential problems have been discovered. Some of the problems are very serious. First it was found that GR is not recovered at large distances if the detailed balance condition is assumed [4, 5]. A “phenomenologically viable” version of the theory without the detailed balance condition was soon introduced [6]. Due to the reduced diffeomorphism symmetry group there is an additional “half” scalar degree of freedom in Hořava-Lifshitz gravity. It has been shown to be strongly coupled at all scales by considering perturbations about a reasonable vacuum [7], regardless whether the detailed balance is assumed or not. This suggests that perturbative GR cannot be reproduced in Hořava-Lifshitz gravity [7], and that the theory could be ruled out by existing observations on the gravitational radiation of binary pulsars, which agree with linearized GR. The low-energy regime of the theory was further analyzed in Ref. [8, 9] where problems with instability and strong coupling of the extra degree of freedom were found. Since then these problems have been confirmed in various papers. Although the non-projectable version of the theory may not give GR as the limit at large distances, some other scenarios, such as the chameleon, may solve this problem. A “healthy extension” of Hořava-Lifshitz gravity was proposed in Ref. [10] that is argued to be free from at least some of the pathologies of the original theory, since the extra scalar mode has a healthy quadratic action. This is achieved by adding terms that involve the spatial 3-vector $N^{-1}\nabla_i N$ into the action. We assume that the Hamiltonian takes the canonical form [see (2.19)] which rules out such terms. Hamiltonian formalism of the healthy extension has been studied in Ref. [11].

Renormalizability of the Hořava-Lifshitz gravity has been investigated beyond the power-counting scheme in Ref. [12].

When the projectability condition is assumed, the theory has a quite simple and consistent Hamiltonian structure. The algebra of constraints was shown to be closed for $z = 1, 2$ in Ref. [2], and this holds for a higher scaling exponent z as well. The Hamiltonian structure of Hořava-Lifshitz gravity without the projectability condition has been

analyzed particularly in Ref. [13, 14]. The non-projectable theory is physically inconsistent for generic couplings [14], including the case with detailed balance [13]. In the case of low-energy effective action a consistent set of constraints can be obtained by imposing an additional constraint ($\pi = 0$) [14, 15, 16].

Recently we proposed the modified $F(R)$ Hořava-Lifshitz gravity [17, 18] that combines the interesting cosmological aspects of $F(R)$ gravity and the possible UV finiteness of Hořava-Lifshitz gravity. In particular, we demonstrated that the solution of spatially-flat FRW equation has two branches: one that coincides with the usual $F(R)$ gravity for a certain choice of parameters, and one that is totally new and typical only for Hořava-Lifshitz gravity. It was shown that unlike to standard Hořava-Lifshitz gravity, our $F(R)$ Hořava-Lifshitz gravity enables the possibility to unify the early-time inflation with the late-time acceleration in accord with the scenario of Ref. [19]. In this paper we present the Hamiltonian analysis of the non-projectable version of this theory, where the lapse N depends also on the spatial coordinates: $N = N(t, \mathbf{x})$. Expectedly the Hamiltonian structure of this theory turns out to be more complex than in the projectable case. Our analysis should be of interest to everyone interested in the Hamiltonian formalism of gravity, and of modified gravity in particular.

2 Hamiltonian analysis

2.1 Action

We assume the ADM decomposition of space-time [20] (for reviews and mathematical background, see [21]). The metric tensor of space-time is decomposed in terms of the ADM variables as

$${}^4g_{\mu\nu}dx^\mu dx^\nu = -(N^2 - N_i N^i)dt^2 + N_i(dt dx^i + dx^i dt) + g_{ij}dx^i dx^j, \quad (2.1)$$

where N is the lapse, N^i is the shift vector, g_{ij} is the spatial metric tensor, and $x^i, i = 1, 2, 3$ are spatial coordinates on the $t = \text{constant}$ hypersurface Σ_t . The covariant derivatives defined by the metric tensors ${}^4g_{\mu\nu}$ and g_{ij} are denoted by $\nabla_\mu^{(4)}$ and ∇_i , respectively. The extrinsic curvature of the hypersurface Σ_t is

$$K_{ij} = \frac{1}{2N} (\dot{g}_{ij} - 2\nabla_{(i} N_{j)}) , \quad (2.2)$$

where the dot denotes the derivative with respect to time t . The scalar associated to the extrinsic curvature is denoted by $K = g^{ij}K_{ij}$. The (intrinsic) curvature of the space Σ_t is defined by the spatial metric g_{ij} in the usual manner. The natural invariant volume element of space-time is decomposed

$$d^4x \sqrt{-{}^4g} = dt d^3\mathbf{x} \sqrt{g} N. \quad (2.3)$$

The action of the non-projectable version of the modified $F(R)$ Hořava-Lifshitz gravity is defined similarly as in the projectable case [17]:

$$S_F = \frac{1}{\kappa^2} \int dt d^3\mathbf{x} \sqrt{g} N F({}^4\tilde{R}), \quad (2.4)$$

$${}^4\tilde{R} \equiv K_{ij}K^{ij} - \lambda K^2 + 2\mu \nabla_\mu^{(4)} (n^\mu \nabla_\nu^{(4)} n^\nu - n^\nu \nabla_\nu^{(4)} n^\mu) - \mathcal{L}_R(g_{ij}).$$

Here λ and μ are constants, n^μ is the unit normal to the spatial hypersurfaces Σ_t , and $\mathcal{L}_R(g_{ij})$ is a function of the three-dimensional metric g_{ij} and the covariant derivatives ∇_i defined by this metric. The crucial difference compared to the theory we proposed and analyzed in Ref. [17] is that the lapse function N does not obey the projectability condition, i.e. it depends also on the spatial coordinates, $N = N(t, \mathbf{x})$. For the Hamiltonian analysis of modified Hořava-Lifshitz-like $F(R)$ gravity, which is a special case of the general projectable theory [17], one can see Ref. [18, 22]. This special case with the further restriction to the parameter value $\mu = 0$ has been proposed and analyzed in Ref. [23]. Yet another special case, given by $F({}^4\tilde{R}) = {}^4\tilde{R}$ and $\mathcal{L}_R(g_{ij}) = -f(R)$, has been studied in Ref. [24].

By introducing two auxiliary fields A and B we can write the action (2.4) into a form that is linear in ${}^4\tilde{R}$:

$$S_F = \frac{1}{\kappa^2} \int dt d^3\mathbf{x} \sqrt{g} N \left[B \left({}^4\tilde{R} - A \right) + F(A) \right]. \quad (2.5)$$

The variation with respect to B yields ${}^4\tilde{R} = A$, which can be inserted back into the action (2.5) in order to produce the original action (2.4). The variation with respect to A yields $B = F'(A)$, where F' denotes the derivative of F with respect to its argument. Thus (2.5) reduces to the action (2.4) when these equations of motion are imposed.

First we rewrite ${}^4\tilde{R}$ in (2.5) in a more explicit and useful form (see (2.4) for the definition of ${}^4\tilde{R}$). The unit normal n^μ to the hypersurface Σ_t in space-time can be written in terms of the lapse and the shift vector as $n^\mu = (n^0, n^i) = \left(\frac{1}{N}, -\frac{N^i}{N} \right)$. The corresponding one-form is $n_\mu = -N \nabla_\mu^{(4)} t = (-N, 0, 0, 0)$. The term in (2.4) that involves the unit normal can be written as

$$\nabla_\mu^{(4)} (n^\mu \nabla_\nu^{(4)} n^\nu - n^\nu \nabla_\nu^{(4)} n^\mu) = \nabla_\mu^{(4)} (n^\mu K) - \frac{1}{N} \Delta N, \quad (2.6)$$

where the spatial Laplacian is $\Delta = g^{ij} \nabla_i \nabla_j$. Thus we can rewrite ${}^4\tilde{R}$

$${}^4\tilde{R} = K_{ij} \mathcal{G}^{ijkl} K_{kl} + 2\mu \nabla_\mu^{(4)} (n^\mu K) - \frac{2\mu}{N} \Delta N - \mathcal{L}_R(g_{ij}), \quad (2.7)$$

where the “generalized De Witt metric” was introduced

$$\mathcal{G}^{ijkl} = \frac{1}{2} (g^{ik} g^{jl} + g^{il} g^{jk}) - \lambda g^{ij} g^{kl}. \quad (2.8)$$

Introducing (2.7) into (2.5) and performing integrations by parts yields the action

$$S_F = \frac{1}{\kappa^2} \int dt d^3\mathbf{x} \sqrt{g} \left\{ N \left[B \left(K_{ij} \mathcal{G}^{ijkl} K_{kl} - \mathcal{L}_R(g_{ij}) - A \right) + F(A) \right] - 2\mu K \left(\dot{B} - N^i \partial_i B \right) - 2\mu N \Delta B \right\}, \quad (2.9)$$

where the integral is taken over the union \mathcal{U} of the $t = \text{constant}$ hypersurfaces Σ_t with t over some interval in \mathbb{R} , and we have written $N n^\mu \nabla_\mu^{(4)} B = \dot{B} - N^i \partial_i B$. We assume that the boundary integrals over $\partial\mathcal{U}$ and $\partial\Sigma_t$ vanish.

2.2 Hamiltonian and momentum constraints

In the Hamiltonian formalism the field variables g_{ij} , N , N_i , A and B have the canonically conjugated momenta π^{ij} , π_N , π^i , π_A and π_B , respectively. Because the action does not

depend on the time derivative of N , N^i or A , their conjugated momenta are the primary constraints:

$$\pi_N(\mathbf{x}) \approx 0, \quad \pi^i(\mathbf{x}) \approx 0, \quad \pi_A(\mathbf{x}) \approx 0. \quad (2.10)$$

For the spatial metric and the field B we have the momenta

$$\pi^{ij} = \frac{\delta S_F}{\delta \dot{g}_{ij}} = \frac{1}{\kappa^2} \sqrt{g} \left[B \mathcal{G}^{ijkl} K_{kl} - \frac{\mu}{N} g^{ij} \left(\dot{B} - N^k \partial_k B \right) \right], \quad (2.11)$$

$$\pi_B = \frac{\delta S_F}{\delta \dot{B}} = -\frac{2\mu}{\kappa^2} \sqrt{g} K. \quad (2.12)$$

We assume $\mu \neq 0$ so that the momentum (2.12) does not vanish. When $\lambda \neq 1/3$, the generalized De Witt metric (2.8) has the inverse

$$\mathcal{G}_{ijkl} = \frac{1}{2} (g_{ik} g_{jl} + g_{il} g_{jk}) - \frac{\lambda}{3\lambda - 1} g_{ij} g_{kl}, \quad \mathcal{G}_{ijkl} \mathcal{G}^{klmn} = \delta_{(i}^{(m} \delta_{j)}^{n)}. \quad (2.13)$$

However, as long as $\mu \neq 0$, the invertibility of (2.8) is not that significant in our theory, because K is given by (2.12), $K = -\frac{\kappa^2}{2\mu\sqrt{g}} \pi_B$, and therefore we have

$$\mathcal{G}^{ijkl} K_{kl} = K^{ij} + \frac{\lambda \kappa^2}{2\mu\sqrt{g}} g^{ij} \pi_B. \quad (2.14)$$

The case $\mu = 0$ will be discussed later in Sec. 2.5.

The Poisson brackets are postulated in the form (equal time t is understood)

$$\begin{aligned} \{g_{ij}(\mathbf{x}), \pi^{kl}(\mathbf{y})\} &= \delta_{(i}^{(k} \delta_{j)}^{l)} \delta(\mathbf{x} - \mathbf{y}), \\ \{N(\mathbf{x}), \pi_N(\mathbf{y})\} &= \delta(\mathbf{x} - \mathbf{y}), \quad \{N_i(\mathbf{x}), \pi^j(\mathbf{y})\} = \delta_i^j \delta(\mathbf{x} - \mathbf{y}), \\ \{A(\mathbf{x}), \pi_A(\mathbf{y})\} &= \delta(\mathbf{x} - \mathbf{y}), \quad \{B(\mathbf{x}), \pi_B(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}). \end{aligned} \quad (2.15)$$

All the other Poisson brackets between the variables vanish. We shall continue to omit the argument (\mathbf{x}) of the fields when there is no risk of confusion.

In the following analysis we may lower and raise spatial indices (i, j, \dots) with the spatial metric g_{ij} and its inverse g^{ij} , e.g. $\pi_{ij} = g_{ik} g_{jl} \pi^{kl}$, and we will denote

$$\pi = g_{ij} \pi^{ij}. \quad (2.16)$$

For some integrals over the space Σ_t , we will routinely perform integration by parts, then apply the divergence form of the Stokes theorem and assume that the resulting boundary integrals over $\partial\Sigma_t$ vanish so that they can be ignored. This can be justified by assuming appropriate boundary conditions for the variables (asymptotic behaviour at infinity), similarly as in GR.

In order to obtain the Hamiltonian, we first solve (2.11)–(2.12) for K_{ij} and \dot{B} ,

$$K_{ij} = \frac{\kappa^2}{\sqrt{g}} \left[\frac{1}{B} \left(\pi_{ij} - \frac{1}{3} g_{ij} \pi \right) - \frac{1}{6\mu} g_{ij} \pi_B \right], \quad (2.17)$$

$$\dot{B} = N^i \partial_i B - N \frac{\kappa^2}{\sqrt{g}} \left(\frac{1}{3\mu} \pi + \frac{1-3\lambda}{6\mu^2} B \pi_B \right), \quad (2.18)$$

and further obtain $\dot{g}_{ij} = 2N K_{ij} + 2\nabla_{(i} N_{j)}$. Therefore both g_{ij} and B are dynamical variables and no more primary constraints are needed. The Hamiltonian is then defined

$$H = \int d^3\mathbf{x} \left(\pi^{ij} \dot{g}_{ij} + \pi_B \dot{B} \right) - L = \int d^3\mathbf{x} \left(N \mathcal{H}_0 + N_i \mathcal{H}^i \right), \quad (2.19)$$

where the Lagrangian L is given by the action (2.9), $S_F = \int dt L$, and the so-called Hamiltonian constraint and the momentum constraints are found to be

$$\begin{aligned}\mathcal{H}_0 &= \frac{\kappa^2}{\sqrt{g}} \left[\frac{1}{B} \left(\pi_{ij} \pi^{ij} - \frac{1}{3} \pi^2 \right) - \frac{1}{3\mu} \pi \pi_B - \frac{1-3\lambda}{12\mu^2} B \pi_B^2 \right] \\ &\quad + \frac{\sqrt{g}}{\kappa^2} [B (\mathcal{L}_R(g_{ij}) + A) - F(A) + 2\mu \Delta B] , \\ \mathcal{H}^i &= -2\nabla_j \pi^{ij} + \nabla^i B \pi_B \\ &= -2\partial_j \pi^{ij} - g^{ij} (2\partial_k g_{jl} - \partial_j g_{kl}) \pi^{kl} + g^{ij} \partial_j B \pi_B ,\end{aligned}\tag{2.20}$$

respectively. We define the total Hamiltonian by

$$H_T = H + \int d^3\mathbf{x} (\lambda_N \pi_N + \lambda_i \pi^i + \lambda_A \pi_A) ,\tag{2.21}$$

where the primary constraints (2.10) are multiplied by the Lagrange multipliers λ_N , λ_i , λ_A . The total Hamiltonian (2.21) generates the time evolution of dynamical variables:

$$\dot{f}(\mathbf{x}) = \{f(\mathbf{x}), H_T\} .\tag{2.22}$$

The primary constraints (2.10) have to be preserved under the time evolution of the system:

$$\begin{aligned}\dot{\pi}_N &= \{\pi_N, H_T\} = -\mathcal{H}_0 , \\ \dot{\pi}^i &= \{\pi^i, H_T\} = -\mathcal{H}^i , \\ \dot{\pi}_A &= \{\pi_A, H_T\} = \frac{\sqrt{g}}{\kappa^2} N (-B + F'(A)) .\end{aligned}\tag{2.23}$$

Therefore we impose the secondary constraints:

$$\begin{aligned}\mathcal{H}_0(\mathbf{x}) &\approx 0 , \quad \mathcal{H}^i(\mathbf{x}) \approx 0 , \\ \Phi_A(\mathbf{x}) &\equiv B(\mathbf{x}) - F'(A(\mathbf{x})) \approx 0 .\end{aligned}\tag{2.24}$$

The Hamiltonian constraint $\mathcal{H}_0(\mathbf{x})$, the momentum constraints $\mathcal{H}^i(\mathbf{x})$ and the constraint $\Phi_A(\mathbf{x})$, are all local. It is convenient to introduce globalized versions of the Hamiltonian and momentum constraints:

$$\begin{aligned}\Phi_0(\eta) &\equiv \int d^3\mathbf{x} \eta \mathcal{H}_0 \approx 0 , \\ \Phi_S(\xi_i) &\equiv \int d^3\mathbf{x} \xi_i \mathcal{H}^i \approx 0 ,\end{aligned}\tag{2.25}$$

where η and ξ_i , $i = 1, 2, 3$ are arbitrary smearing functions that vanish rapidly enough at infinity — the choices $\eta = \delta(\mathbf{x} - \mathbf{y})$ and $\xi_i = \delta_i^j \delta(\mathbf{x} - \mathbf{y})$ will produce the local constraints \mathcal{H}_0 and \mathcal{H}^j , which in turn imply the smeared constraints.

2.3 Consistency of the secondary constraints under dynamics

The total Hamiltonian (2.21) can be rewritten in terms of the Hamiltonian and momentum constraints (2.25) as

$$H_T = \Phi_0(N) + \Phi_S(N_i) + \int d^3\mathbf{x} (\lambda_N \pi_N + \lambda_i \pi^i + \lambda_A \pi_A) .\tag{2.26}$$

The consistency of the system requires that also the secondary constraints $\Phi_0(\eta)$, $\Phi_S(\xi_i)$ and $\Phi_A(\mathbf{x})$ have to be preserved under time evolution generated by the total Hamiltonian (2.26):

$$\begin{aligned}\dot{\Phi}_0(\eta) &= \{\Phi_0(\eta), \Phi_0(N)\} + \{\Phi_0(\eta), \Phi_S(N_i)\} + \int d^3\mathbf{x} \lambda_A(\mathbf{x}) \{\Phi_0(\eta), \pi_A(\mathbf{x})\} \approx 0, \\ \dot{\Phi}_S(\xi_i) &= \{\Phi_S(\xi_i), \Phi_0(N)\} + \{\Phi_S(\xi_i), \Phi_S(N_i)\} \approx 0, \\ \dot{\Phi}_A(\mathbf{x}) &= \{\Phi_A(\mathbf{x}), \Phi_0(N)\} + \{\Phi_A(\mathbf{x}), \Phi_S(N_i)\} + \int d^3\mathbf{y} \lambda_A(\mathbf{y}) \{\Phi_A(\mathbf{x}), \pi_A(\mathbf{y})\} \approx 0,\end{aligned}\tag{2.27}$$

where we have used the fact that the constraints π_N and π^i have strongly vanishing Poisson brackets with every constraint, and that

$$\{\Phi_S(\xi_i), \pi_A\} = 0.\tag{2.28}$$

We need to calculate the rest of the algebra of the constraints under the Poisson bracket. The Poisson brackets between the momentum constraint $\Phi_S(\xi_i)$ and the canonical variables are

$$\begin{aligned}\{\Phi_S(\xi_i), B\} &= -\xi^i \partial_i B, \\ \{\Phi_S(\xi_i), \pi_B\} &= -\partial_i (\xi^i \pi_B), \\ \{\Phi_S(\xi_k), g_{ij}\} &= -\xi^k \partial_k g_{ij} - g_{ik} \partial_j \xi^k - g_{jk} \partial_i \xi^k, \\ \{\Phi_S(\xi_k), \pi^{ij}\} &= -\partial_k (\xi^k \pi^{ij}) + \pi^{ik} \partial_k \xi^j + \pi^{jk} \partial_k \xi^i,\end{aligned}\tag{2.29}$$

and trivially zero for A and π_A ,

$$\{\Phi_S(\xi_i), A\} = 0, \quad \{\Phi_S(\xi_i), \pi_A\} = 0.\tag{2.30}$$

Thus $\Phi_S(\xi_i)$ generates the spatial diffeomorphisms for the variables $B, \pi_B, g_{ij}, \pi^{ij}$, and consequently for any function or functional constructed from these variables, and treats the variables A, π_A as constants. By using this result (2.29)–(2.30) we obtain the Lie algebra of the generators $\Phi_S(\xi_i)$:

$$\{\Phi_S(\xi_i), \Phi_S(\eta_i)\} = \Phi_S(\xi^j \partial_j \eta_i - \eta^j \partial_j \xi_i) \approx 0.\tag{2.31}$$

Then we calculate their Poisson brackets with the Hamiltonian constraint $\Phi_0(\eta)$:

$$\{\Phi_S(\xi_i), \Phi_0(\eta)\} = \Phi_0(\xi^i \partial_i \eta) \approx 0,\tag{2.32}$$

which tells us that \mathcal{H}_0 is a scalar density under spatial diffeomorphism. The momentum constraints $\mathcal{H}^i(\mathbf{x})$ will be first-class everywhere and consistent under time evolution.

Then consider the constraint $\Phi_A(\mathbf{x})$, whose nonvanishing Poisson brackets are

$$\begin{aligned}\{\Phi_A(\mathbf{x}), \Phi_0(\eta)\} &= -\eta \frac{\kappa^2}{\sqrt{g}} \left(\frac{1}{3\mu} \pi + \frac{1-3\lambda}{6\mu^2} B \pi_B \right), \\ \{\Phi_A(\mathbf{x}), \Phi_S(\xi_i)\} &= \xi^i \partial_i B, \\ \{\Phi_A(\mathbf{x}), \pi_A(\mathbf{y})\} &= -F''(A(\mathbf{x})) \delta(\mathbf{x} - \mathbf{y}).\end{aligned}\tag{2.33}$$

Thus, in order to satisfy the consistency conditions (2.27), we have to impose the tertiary constraint

$$N^i \partial_i B - N \frac{\kappa^2}{\sqrt{g}} \left(\frac{1}{3\mu} \pi + \frac{1-3\lambda}{6\mu^2} B \pi_B \right) - \lambda_A F''(A) \approx 0.\tag{2.34}$$

Since $F''(A) = 0$ would essentially reproduce the usual non-projectable Hořava-Lifshitz gravity, we assume that $F''(A) \neq 0$. The first two terms in (2.34), i.e. the expression for \dot{B} in (2.18), does not vanish due to the established constraints (2.10) and (2.24). Therefore (2.34) is a restriction on the Lagrange multiplier λ_A , and we can solve it:

$$\lambda_A = \frac{1}{F''(A)} \left(N^i \partial_i B - N \frac{\kappa^2}{\sqrt{g}} \left(\frac{1}{3\mu} \pi + \frac{1-3\lambda}{6\mu^2} B \pi_B \right) \right). \quad (2.35)$$

Introducing (2.35) into the Hamiltonian (2.26) ensures that the constraint $\Phi_A(\mathbf{x})$ is consistent.

Finally we consider the Hamiltonian constraint $\Phi_0(\eta)$. The Poisson bracket with the primary constraint $\pi_A(\mathbf{x})$ vanishes in (2.27)

$$\{\Phi_0(\eta), \pi_A(\mathbf{x})\} = \eta \frac{\sqrt{g}}{\kappa^2} \Phi_A(\mathbf{x}) \approx 0. \quad (2.36)$$

Also the Poisson bracket of the Hamiltonian constraint $\Phi_0(\eta)$ with itself has to be calculated in order to check its consistency. Since the Poisson bracket is antisymmetric with respect to its arguments and the Hamiltonian constraint does not depend on the primary constraints (2.10), we can write the Poisson bracket of Hamiltonian constraints as

$$\{\Phi_0(\xi), \Phi_0(\eta)\} = \int d^3 \mathbf{x} \left(\frac{\delta \Phi_0(\xi)}{\delta g_{ij}(\mathbf{x})} \frac{\delta \Phi_0(\eta)}{\delta \pi^{ij}(\mathbf{x})} + \frac{\delta \Phi_0(\xi)}{\delta B(\mathbf{x})} \frac{\delta \Phi_0(\eta)}{\delta \pi_B(\mathbf{x})} \right) - (\xi \leftrightarrow \eta) \quad (2.37)$$

or

$$\begin{aligned} \{\Phi_0(\xi), \Phi_0(\eta)\} &= \int d^3 \mathbf{x} \left(\{\Phi_0(\xi), \pi^{ij}(\mathbf{x})\} \{g_{ij}(\mathbf{x}), \Phi_0(\eta)\} \right. \\ &\quad \left. + \{\Phi_0(\xi), \pi_B(\mathbf{x})\} \{B(\mathbf{x}), \Phi_0(\eta)\} \right) - (\xi \leftrightarrow \eta). \end{aligned} \quad (2.38)$$

Further simplification follows from the fact that the Poisson brackets of the Hamiltonian constraint $\Phi_0(\eta)$ with the field variables g_{ij} and B are proportional to η , i.e. of the form ηf where f is a function of the canonical variables:

$$\{g_{ij}(\mathbf{x}), \Phi_0(\eta)\} = \eta \frac{\kappa^2}{\sqrt{g}} \left[\frac{2}{B} \left(\pi_{ij} - \frac{1}{3} g_{ij} \pi \right) - \frac{1}{3\mu} g_{ij} \pi_B \right] = 2\eta K_{ij}, \quad (2.39)$$

$$\{B(\mathbf{x}), \Phi_0(\eta)\} = -\eta \frac{\kappa^2}{\sqrt{g}} \left(\frac{1}{3\mu} \pi + \frac{1-3\lambda}{6\mu^2} B \pi_B \right). \quad (2.40)$$

Therefore, the parts of the Poisson brackets of the Hamiltonian constraint $\Phi_0(\xi)$ with the momenta π^{ij} and π_B that are proportional to ξ will contribute to terms that are proportional to $\xi\eta = \eta\xi$. These terms will necessarily cancel out due to the antisymmetry of (2.38) under $\xi \leftrightarrow \eta$. Thus, in order to calculate (2.38), we only need those parts of the Poisson brackets with the momenta that contain spatial derivatives of ξ . These Poisson brackets are found to be

$$\begin{aligned} \{\Phi_0(\xi), \pi^{ij}(\mathbf{x})\} &= -\xi \frac{\kappa^2}{\sqrt{g}} g^{ij} \left[\frac{1}{2B} \left(\pi_{kl} \pi^{kl} - \frac{1}{3} \pi^2 \right) - \frac{1}{6\mu} \pi \pi_B - \frac{1-3\lambda}{24\mu^2} B \pi_B^2 \right] \\ &\quad + \xi \frac{\sqrt{g}}{\kappa^2} g^{ij} \left[\frac{1}{2} B (\mathcal{L}_R(g_{kl}) + A) - \frac{1}{2} F(A) + \mu \Delta B \right] \\ &\quad + \xi \frac{\kappa^2}{\sqrt{g}} \left[\frac{2}{B} \left(\pi^i{}_k \pi^{jk} - \frac{1}{3} \pi^{ij} \pi \right) - \frac{1}{3\mu} \pi^{ij} \pi_B \right] \\ &\quad + \frac{1}{\kappa^2} \int d^3 \mathbf{y} \sqrt{g} \xi(\mathbf{y}) \left(B(\mathbf{y}) \frac{\delta \mathcal{L}_R(g_{kl}(\mathbf{y}))}{\delta g_{ij}(\mathbf{x})} + 2\mu \frac{\delta \Delta B(\mathbf{y})}{\delta g_{ij}(\mathbf{x})} \right) \end{aligned} \quad (2.41)$$

and

$$\begin{aligned}\{\Phi_0(\xi), \pi_B(\mathbf{x})\} &= -\xi \frac{\kappa^2}{\sqrt{g}} \left[\frac{1}{B^2} \left(\pi_{ij} \pi^{ij} - \frac{1}{3} \pi^2 \right) + \frac{1-3\lambda}{12\mu^2} \pi_B^2 \right] \\ &+ \xi \frac{\sqrt{g}}{\kappa^2} (\mathcal{L}_R(g_{ij}) + A) + \frac{\sqrt{g}}{\kappa^2} 2\mu \Delta \xi, \end{aligned} \quad (2.42)$$

where the last term was obtained from

$$\left\{ \int d^3 \mathbf{y} \xi \frac{\sqrt{g}}{\kappa^2} 2\mu \Delta B, \pi_B(\mathbf{x}) \right\} = \frac{2\mu}{\kappa^2} \int d^3 \mathbf{y} \sqrt{g} \xi(\mathbf{y}) \frac{\delta \Delta B(\mathbf{y})}{\delta B(\mathbf{x})} = \frac{\sqrt{g}}{\kappa^2} 2\mu \Delta \xi(\mathbf{x}). \quad (2.43)$$

We find immediately that only the last term in both (2.41) and (2.42) give nonzero contributions to the Poisson bracket (2.38).

Then we must calculate the variations of the potential part $\mathcal{L}_R(g_{ij})$ and ΔB with respect to the spatial metric in (2.41). For this we need the variations of the involved geometric quantities. The variations of the connection coefficients Γ_{ij}^k , the Ricci tensor R_{ij} and the scalar curvature R are given by

$$\begin{aligned}\delta \Gamma_{ij}^k &= \frac{1}{2} g^{kl} (\nabla_i \delta g_{lj} + \nabla_j \delta g_{il} - \nabla_l \delta g_{ij}), \\ \delta R_{ij} &= \frac{1}{2} g^{kl} (\nabla_k \nabla_i \delta g_{lj} + \nabla_k \nabla_j \delta g_{li} - \nabla_i \nabla_j \delta g_{kl}) - \frac{1}{2} \Delta \delta g_{ij}, \\ \delta R &= -R^{ij} \delta g_{ij} + \nabla^i \nabla^j \delta g_{ij} - g^{ij} \Delta \delta g_{ij}. \end{aligned} \quad (2.44)$$

For ΔB we obtain the variation

$$\delta \Delta B = -\delta g_{ij} \nabla^i \nabla^j B + \frac{1}{2} g^{ij} (\nabla_i \delta g_{kj} + \nabla_j \delta g_{ik} - \nabla_k \delta g_{ij}) \nabla^k B \quad (2.45)$$

and then in (2.41) we have

$$\begin{aligned}a_{ij}(\mathbf{x}) \int d^3 \mathbf{y} \sqrt{g} \xi(\mathbf{y}) 2\mu \frac{\delta \Delta B(\mathbf{y})}{\delta g_{ij}(\mathbf{x})} &= a_{ij} 2\mu \sqrt{g} \left[-2\xi \nabla^{(i} \nabla^{j)} B - \nabla^{(i} \xi \nabla^{j)} B \right. \\ &\quad \left. + \frac{1}{2} g^{ij} (\nabla_k \xi \nabla^k B + \xi \Delta B) \right], \end{aligned} \quad (2.46)$$

where a_{ij} represents any tensor, and the first term and the last term will be cancelled for the familiar reason. Thus we obtain the general result for the Poisson bracket of Hamiltonian constraints (2.38):

$$\begin{aligned}\{\Phi_0(\xi), \Phi_0(\eta)\} &= \int d^3 \mathbf{x} \eta \left\{ \left[\frac{2}{B} \left(\pi_{ij} - \frac{1}{3} g_{ij} \pi \right) - \frac{1}{3\mu} g_{ij} \pi_B \right] (\mu g^{ij} \nabla^k B \nabla_k \xi \right. \\ &\quad \left. - 2\mu \nabla^{(i} B \nabla^{j)} \xi + \frac{1}{\sqrt{g}} \int d^3 \mathbf{y} \sqrt{g} \xi(\mathbf{y}) B(\mathbf{y}) \frac{\delta \mathcal{L}_R(g_{kl}(\mathbf{y}))}{\delta g_{ij}(\mathbf{x})} \right) \\ &\quad \left. - \left(\frac{2}{3} \pi + \frac{1-3\lambda}{3\mu} B \pi_B \right) \Delta \xi \right\} - (\xi \leftrightarrow \eta). \end{aligned} \quad (2.47)$$

Further progress can be obtained by specifying the form of the potential part $\mathcal{L}_R(g_{ij})$ in the action.

For $\mathcal{L}_R(g_{ij})$ let us start with the simplest case that could be the low-energy effective potential

$$\mathcal{L}_R(g_{ij}) = \alpha_0 + \alpha_1 R, \quad (2.48)$$

where α_0 and α_1 are coupling constants. (2.48) gives ${}^4\tilde{R}$ in (2.7) as a “deformed scalar curvature + constant”:

$${}^4\tilde{R} = K^{ij}K_{ij} - \lambda K^2 - \alpha_1 R + 2\mu \nabla_\mu^{(4)}(n^\mu K) - \frac{2\mu}{N} \Delta N - \alpha_0. \quad (2.49)$$

When $\lambda = 1$, $\mu = 1$, $\alpha_1 = -1$ and $\alpha_0 = 0$, this ${}^4\tilde{R}$ reduces to the scalar curvature 4R of space-time. For asymptotically flat spaces we assume $\alpha_0 = 0$. For (2.48) we obtain

$$a_{ij}(\mathbf{x}) \int d^3\mathbf{y} \sqrt{g} \xi(\mathbf{y}) B(\mathbf{y}) \frac{\delta \mathcal{L}_R(g_{kl}(\mathbf{y}))}{\delta g_{ij}(\mathbf{x})} = a_{ij} \alpha_1 \sqrt{g} \left[-\xi B R^{ij} + \nabla^{(i} \nabla^{j)} (\xi B) - g^{ij} \Delta (\xi B) \right], \quad (2.50)$$

where a_{ij} represents any tensor, and again the parts that contain no derivatives of ξ will cancel out in the Poisson bracket (2.47). This expression contains first and second order spatial derivatives of ξ . Thus we obtain the result

$$\{\Phi_0(\xi), \Phi_0(\eta)\} = \int d^3\mathbf{x} \eta \left(C_2^{ij} \nabla_{(i} \nabla_{j)} \xi + C_1^i \nabla_i \xi \right) - (\xi \leftrightarrow \eta), \quad (2.51)$$

where we have defined:

$$\begin{aligned} C_2^{ij} &= 2\alpha_1 \pi^{ij} - g^{ij} \left(\frac{2(\alpha_1 + 1)}{3} \pi + \frac{1 - 2\alpha_1 - 3\lambda}{3\mu} B \pi_B \right), \\ C_1^i &= \left[4(\alpha_1 - \mu) \pi^{ij} + g^{ij} \left(\frac{4(-\alpha_1 + \mu)}{3} \pi + \frac{4\alpha_1 - \mu}{3\mu} B \pi_B \right) \right] \frac{\nabla_j B}{B}. \end{aligned} \quad (2.52)$$

The tensor C_2^{ij} is symmetric and so is the part of C_1^i that multiplies $\nabla_j B$ in the definition (2.52). Note that in the case $\alpha_1 = -1$, $\lambda = 1$ the coefficient of the $\Delta \xi$ -term vanishes in (2.51) so that $C_2^{ij} \rightarrow 2\alpha_1 \pi^{ij}$. Finally we can write (2.51) into a form where the integrand is proportional to ξ :

$$\{\Phi_0(\xi), \Phi_0(\eta)\} = \int d^3\mathbf{x} \xi \left(E_1^i \nabla_i \eta + E_0 \eta \right). \quad (2.53)$$

where we have defined

$$\begin{aligned} E_1^i &= 2 \left(\nabla_j C_2^{ij} - C_1^i \right), \\ E_0 &= \nabla_{(i} \nabla_{j)} C_2^{ij} - \nabla_i C_1^i = \frac{1}{2} \nabla_i E_1^i. \end{aligned} \quad (2.54)$$

Note that the terms involving the second spatial derivative $\nabla_{(i} \nabla_{j)} \eta$ cancel each other independently of the form of C_2^{ij} . Now the condition that is necessary for the preservation of the local Hamiltonian constraint $\mathcal{H}_0(\mathbf{x})$ can be obtained by inserting $\xi = \delta(\mathbf{x} - \mathbf{y})$ and $\eta = N$ into the Poisson bracket (2.53). Thus, in order to ensure the consistency of the Hamiltonian constraint $\mathcal{H}_0(\mathbf{x})$, we must impose the constraint

$$\tilde{E}_1^i \nabla_i N + \tilde{E}_0 N \approx 0, \quad (2.55)$$

where we have factored out some parts of the E_n 's that vanish due to the momentum constraints by defining:

$$\begin{aligned} \tilde{E}_1^i &= E_1^i + 2\alpha_1 \mathcal{H}^i \\ &= 2 \left[-\frac{2(\alpha_1 + 1)}{3} \nabla^i \pi + 4(-\alpha_1 + \mu) \left(\pi^{ij} - \frac{1}{3} g^{ij} \pi \right) \frac{\nabla_j B}{B} \right. \\ &\quad \left. - \frac{1 - 2\alpha_1 - 3\lambda}{3\mu} B \nabla^i \pi_B + \left(\frac{3\lambda + \mu - 2\alpha_1 - 1}{3\mu} + \alpha_1 \right) \nabla^i B \pi_B \right], \\ \tilde{E}_0 &= E_0 + \alpha_1 \nabla_i \mathcal{H}^i = \frac{1}{2} \nabla_i \tilde{E}_1^i. \end{aligned} \quad (2.56)$$

This is a homogeneous first-order partial differential equation for the lapse N . As such it always has the solution $N = 0$. Due to the relation (2.54) of E_0 and E_1^i , we can rewrite the constraint (2.55) as a divergence (after multiplying it by $2N$)

$$\nabla_i \left(N^2 \tilde{E}_1^i \right) \approx 0. \quad (2.57)$$

Let us next consider the case with the critical exponent $z = 3$ that could provide an UV complete theory. For the potential $\mathcal{L}_R(g_{ij})$ there are many terms that have the same scaling dimension as the kinetic terms under the anisotropic scaling (1.1) with $z = 3$. Such terms are, for example, the terms quadratic in curvature

$$\nabla^i R \nabla_i R, \quad \nabla^i R^{jk} \nabla_i R_{jk}, \quad \nabla^i R^{jk} \nabla_j R_{ki} \quad (2.58)$$

and

$$R \Delta R, \quad R^{ij} \Delta R_{ij}, \quad R^{ij} \nabla_i \nabla_j R, \quad (2.59)$$

which modify the propagator in addition to providing interactions, and the terms cubic in curvature that are pure interactions

$$R^3, \quad R R^{ij} R_{ij}, \quad R^i{}_j R^j{}_k R^k{}_i. \quad (2.60)$$

Many of the terms are related to each other due to the properties of the Riemann tensor in three dimensions (Weyl tensor vanishes), the Bianchi identity, and integration by parts. Indeed only two terms of the types (2.58) and (2.59) are independent, which we choose to be

$$\nabla^i R^{jk} \nabla_i R_{jk}, \quad R \Delta R. \quad (2.61)$$

The most general potential that contains all the independent renormalizable and super-renormalizable terms, while maintaining the canonical form of the Hamiltonian (2.19), is

$$\begin{aligned} \mathcal{L}_R(g_{ij}) = & \alpha_0 + \alpha_1 R + \alpha_2 R^2 + \alpha_3 R^{ij} R_{ij} + \alpha_4 R^3 + \alpha_5 R R^{ij} R_{ij} \\ & + \alpha_6 R^i{}_j R^j{}_k R^k{}_i + \alpha_7 \nabla^i R^{jk} \nabla_i R_{jk} + \alpha_8 R \Delta R. \end{aligned} \quad (2.62)$$

Under the variation of the spatial metric the terms involving second order spatial derivatives of curvature (2.59) contain spatial derivatives of the variation δg_{ij} up to fourth order. Likewise, the variations of the terms (2.58) contain spatial derivatives of the variation δg_{ij} up to third order, and the variation of (2.60) up to second order. This means that the variation of the potential (2.62) contains spatial derivatives of the variation δg_{ij} up to fourth order. Thus the Poisson bracket of Hamiltonian constraints has the form

$$\{\Phi_0(\xi), \Phi_0(\eta)\} = \int d^3 \mathbf{x} \eta \left(C_4^{ijkl} \nabla_{ijkl} \xi + C_3^{ijk} \nabla_{ijk} \xi + C_2^{ij} \nabla_{ij} \xi + C_1^i \nabla_i \xi \right) - (\xi \leftrightarrow \eta), \quad (2.63)$$

where C_n 's are symmetric tensors consisting of the canonical variables and their spatial derivatives, and we denote $\nabla_{ij} = \nabla_{(i} \nabla_{j)}$ etc. For arbitrary couplings α_m the tensors C_n have quite complicated forms, which we do not present here. After integration by parts we obtain

$$\{\Phi_0(\xi), \Phi_0(\eta)\} = \int d^3 \mathbf{x} \xi \left(E_3^{ijk} \nabla_{ijk} \eta + E_2^{ij} \nabla_{ij} \eta + E_1^i \nabla_i \eta + E_0 \eta \right), \quad (2.64)$$

where we have defined

$$\begin{aligned}
E_3^{ijk} &= 4\nabla_i C_4^{ijkl} - 2C_3^{ijk}, \\
E_2^{ij} &= 6\nabla_{kl} C_4^{ijkl} - 3\nabla_k C_3^{ijk} = \frac{3}{2}\nabla_k E_3^{ijk}, \\
E_1^i &= 4\nabla_{jkl} C_4^{ijkl} - 3\nabla_{jk} C_3^{ijk} + 2\nabla_j C_2^{ij} - 2C_1^i, \\
E_0 &= \nabla_{ijkl} C_4^{ijkl} - \nabla_{ijk} C_3^{ijk} + \nabla_{ij} C_2^{ij} - \nabla_i C_1^i.
\end{aligned} \tag{2.65}$$

The resulting tertiary constraint is of the form

$$E_3^{ijk}\nabla_{ijk}N + E_2^{ij}\nabla_{ij}N + E_1^i\nabla_iN + E_0N \approx 0. \tag{2.66}$$

The condition (2.66) is again homogeneous in N and is therefore satisfied by $N = 0$. Note that the fourth order spatial derivative of N cancels out similarly as the second order derivative did in the case (2.48). More generally for every even n , C_n contributes to E_k for $k < n$, but never to E_n . For every uneven n , the contribution of C_n to E_n is $-2C_n$. Thus the highest spatial derivative of N in the constraints like (2.55) and (2.66) is always of uneven order. This is clearly a very general result that holds in any theory where the Poisson bracket of Hamiltonian constraints is of the form (2.63) with nonvanishing coefficients C_n usually up to $n = z + 1$ — including the usual Hořava-Lifshitz gravity. The constraints (2.57) and (2.66) will be discussed further in Sec. 2.7.

2.4 Elimination of the auxiliary field A

According to the Poisson brackets between the constraints (2.31)–(2.33) and (2.36), we can set the second-class constraints $\pi_A(\mathbf{x})$ and $\Phi_A(\mathbf{x})$ to vanish strongly. Then the Hamiltonian constraint $\Phi_0(\eta)$ will have weakly vanishing Poisson brackets with every constraint except with itself (2.64). The momentum constraints $\Phi_S(\xi_i)$ will be first-class assuming its Poisson bracket with possible additional constraints are weakly vanishing too. To this end, we replace the Poisson bracket with the Dirac bracket, which is given by

$$\begin{aligned}
\{f(\mathbf{x}), h(\mathbf{y})\}_{\text{DB}} &= \{f(\mathbf{x}), h(\mathbf{y})\} + \int d^3z \frac{1}{F''(A(\mathbf{z}))} (\{f(\mathbf{x}), \pi_A(\mathbf{z})\}\{\Phi_A(\mathbf{z}), h(\mathbf{y})\} \\
&\quad - \{f(\mathbf{x}), \Phi_A(\mathbf{z})\}\{\pi_A(\mathbf{z}), h(\mathbf{y})\}), \tag{2.67}
\end{aligned}$$

where f and h are any functions of the canonical variables. Assuming we can solve the constraint $\Phi_A(\mathbf{x}) = 0$, i.e. $B = F'(A)$, for $A = \tilde{A}(B)$, where \tilde{A} is the inverse of the function F' , we can eliminate the variables A and π_A . Thus the final variables of the system are $g_{ij}, \pi^{ij}, B, \pi_B$. The lapse N and the shift vector N_i , together with λ_N and λ_i , are non-dynamical multipliers. Since every dynamical variable has a vanishing Poisson bracket with the constraint π_A , the Dirac bracket (2.67) reduces to the Poisson bracket,

$$\{f(\mathbf{x}), h(\mathbf{y})\}_{\text{DB}} = \{f(\mathbf{x}), h(\mathbf{y})\}. \tag{2.68}$$

2.5 The cases $F'' = 0$ and $\mu = 0$

For $F'' = 0$, i.e. $F(4\tilde{R}) = c_1 4\tilde{R} + c_0$ with constants c_1 and c_0 , the action (2.4) with (2.62) reduces to the action of the usual Hořava-Lifshitz gravity without the detailed balance condition, when the couplings are redefined: $c_1/\kappa^2 \rightarrow 1/\kappa^2$, $(c_1\alpha_0 - c_0)/\kappa^2 \rightarrow \alpha_0$ and $c_1\alpha_n/\kappa^2 \rightarrow \alpha_n$ for $n = 1, 2, \dots, 8$. The value of the parameter μ is irrelevant, since the term

involving (2.6) is a total divergence and hence it can be dropped. For Hamiltonian analysis of non-projectable Hořava-Lifshitz gravity see Ref. [14] and references therein, recalling the observations about the general structure of the Poisson bracket of Hamiltonian constraints made in Sec. 2.3.

Let us then consider the case $\mu = 0$, and assume $F'' \neq 0$. As we mentioned this is a generalization of the (second) model proposed in Ref. [23] without the detailed balance condition. The additional primary constraint $\pi_B(\mathbf{x}) \approx 0$ has to be introduced in addition to (2.10). The canonical momenta are

$$\pi^{ij} = \frac{1}{\kappa^2} \sqrt{g} B \mathcal{G}^{ijkl} K_{kl}. \quad (2.69)$$

The field B is non-dynamical. For $\lambda \neq 1/3$, we obtain the Hamiltonian and the momentum constraints

$$\begin{aligned} \mathcal{H}_0 &= \frac{\kappa^2}{\sqrt{g} B} \pi^{ij} \mathcal{G}_{ijkl} \pi^{kl} + \frac{\sqrt{g}}{\kappa^2} [B (\mathcal{L}_R(g_{ij}) + A) - F(A)], \\ \mathcal{H}^i &= -2 \nabla_j \pi^{ij}. \end{aligned} \quad (2.70)$$

The preservation of the constraint $\pi_B(\mathbf{x})$ under time evolution implies the secondary constraint

$$\Phi_B(\mathbf{x}) \equiv \frac{\kappa^2}{\sqrt{g} B^2} \pi^{ij} \mathcal{G}_{ijkl} \pi^{kl} - \frac{\sqrt{g}}{\kappa^2} (\mathcal{L}_R(g_{ij}) + A) \approx 0. \quad (2.71)$$

The constraints $\chi_a(\mathbf{x}) = (\pi_A, \Phi_A, \pi_B, \Phi_B)$ are second-class and the corresponding Dirac bracket is defined

$$\{f(\mathbf{x}), h(\mathbf{y})\}_{\text{DB}} = \{f(\mathbf{x}), h(\mathbf{y})\} - \int d^3 z d^3 z' \{f(\mathbf{x}), \chi_a(\mathbf{z})\} C^{ab}(\mathbf{z}, \mathbf{z}') \{\chi_b(\mathbf{z}'), h(\mathbf{y})\} \quad (2.72)$$

where $C^{ab}(\mathbf{z}, \mathbf{z}') = C^{ab}(\mathbf{z}) \delta(\mathbf{z} - \mathbf{z}')$ is the inverse of

$$C_{ab}(\mathbf{x}, \mathbf{y}) \equiv \{\chi_a(\mathbf{x}), \chi_b(\mathbf{y})\} = C_{ab}(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}).$$

The nonvanishing components of these antisymmetric matrices are

$$\begin{aligned} C_{12}(\mathbf{x}) &= F''(A), \quad C_{14}(\mathbf{x}) = \frac{\sqrt{g}}{\kappa^2}, \\ C_{23}(\mathbf{x}) &= 1, \quad C_{34}(\mathbf{x}) = \frac{2\kappa^2}{\sqrt{g} B^3} \pi^{ij} \mathcal{G}_{ijkl} \pi^{kl}, \end{aligned} \quad (2.73)$$

and

$$\begin{aligned} C^{12}(\mathbf{z}) &= -\frac{2\pi^{ij} \mathcal{G}_{ijkl} \pi^{kl}}{2\pi^{ij} \mathcal{G}_{ijkl} \pi^{kl} F''(A) + g B^3 / \kappa^4}, \\ C^{14}(\mathbf{z}) &= -\frac{\sqrt{g} B^3 / \kappa^2}{2\pi^{ij} \mathcal{G}_{ijkl} \pi^{kl} F''(A) + g B^3 / \kappa^4}, \\ C^{23}(\mathbf{z}) &= -\frac{g B^3 / \kappa^4}{2\pi^{ij} \mathcal{G}_{ijkl} \pi^{kl} F''(A) + g B^3 / \kappa^4}, \\ C^{34}(\mathbf{z}) &= -\frac{\sqrt{g} F''(A) B^3 / \kappa^2}{2\pi^{ij} \mathcal{G}_{ijkl} \pi^{kl} F''(A) + g B^3 / \kappa^4}. \end{aligned} \quad (2.74)$$

Since the constraints $\chi_1 = \pi_A$ and $\chi_3 = \pi_B$ have vanishing Poisson brackets with the canonical variables g_{ij} and π^{ij} , and we have $C^{24}(\mathbf{x}) = 0$, we again find that the Dirac bracket reduces to the Poisson bracket (2.68) for any functions of g_{ij} and π^{ij} . The auxiliary fields can be eliminated from the action by solving $\Phi_A(\mathbf{x}) = 0$ and $\Phi_B(\mathbf{x}) = 0$ for

$$A = \tilde{A} \left(\frac{\kappa^2}{\sqrt{g}} \pi^{ij} \mathcal{G}_{ijkl} \pi^{kl}, \frac{\sqrt{g}}{\kappa^2} \mathcal{L}_R(g_{ij}) \right), \quad B = F'(\tilde{A}), \quad (2.75)$$

where \tilde{A} satisfies

$$\frac{\kappa^2}{\sqrt{g} F'(\tilde{A})^2} \pi^{ij} \mathcal{G}_{ijkl} \pi^{kl} - \frac{\sqrt{g}}{\kappa^2} \left(\mathcal{L}_R(g_{ij}) + \tilde{A} \right) = 0. \quad (2.76)$$

Then the Hamiltonian constraint can be written in terms of g_{ij} and π^{ij} :

$$\mathcal{H}_0 = \frac{2\kappa^2}{\sqrt{g} F'(\tilde{A})} \pi^{ij} \mathcal{G}_{ijkl} \pi^{kl} - \frac{\sqrt{g}}{\kappa^2} F(\tilde{A}). \quad (2.77)$$

The seemingly simple form of this Hamiltonian can be deceiving, since the form of the function \tilde{A} can be complex. For example, when $F(R)$ is a polynomial of degree n , then \tilde{A} is a root of a polynomial $p(\tilde{A})$ of degree $2(n-1) + 1$.

The momentum constraints (2.70) clearly generate the spatial diffeomorphisms of the canonical variables, since \mathcal{H}^i are identical to the momentum constraints of GR. It is also clear that the Poisson bracket of Hamiltonian constraints has the general form (2.63) when the potential takes the general canonical form (2.62).

Since $\mathcal{G}^{ijkl} g_{kl} = (1 - 3\lambda) g^{ij}$, the value $\lambda = 1/3$ implies the primary constraint $\pi(\mathbf{x}) \approx 0$ similarly as in the usual Hořava-Lifshitz gravity. Then the Hamiltonian and the momentum constraints are found to be as in (2.70) but with $\pi^{ij} \mathcal{G}_{ijkl} \pi^{kl}$ replaced by $\pi_{ij} \pi^{ij}$. The preservation of $\pi(\mathbf{x}) \approx 0$ under time evolution is a nontrivial matter that is not discussed here.

2.6 On the class of theories with a projectable lapse N

For comparison we briefly consider the class of theories with a projectable lapse $N = N(t)$ (see [17] for more details). In this case there is only one (global) Hamiltonian constraint $\Phi_0 \equiv \int d^3 \mathbf{x} \mathcal{H}_0 = \Phi_0(1)$. Now the Poisson bracket of Hamiltonian constraints vanishes,

$$\{\Phi_0, \Phi_0\} = 0, \quad (2.78)$$

and there is no need for additional constraints like the one (2.66) in the class of theories with a non-projectable lapse.

As explained in Sec. 2.4, the second-class constraints $\pi_A(\mathbf{x})$ and $\Phi_A(\mathbf{x})$ can be set to vanish strongly by utilizing the Dirac bracket (2.67), which reduces to the Poisson bracket (2.68). As a result the Hamiltonian constraint Φ_0 and the momentum constraints $\Phi_S(\xi_i)$ are first-class, since they have vanishing Dirac brackets with every constraint. Finally the total Hamiltonian is the sum of the first-class constraints

$$H_T = N\Phi_0 + \Phi_S(N_i) + \lambda_N \pi_N + \int d^3 \mathbf{x} \lambda_i \pi^i. \quad (2.79)$$

In the case $\mu = 0$ discussed in Sec. 2.5, the projectability condition has the exact same effect as in the case $\mu \neq 0$ considered above.

2.7 Interpretation and analysis of the tertiary constraints — conditions on the lapse N or not?

Let us discuss the conditions (2.57) and (2.66). In order to obtain real dynamics, the requirement $N \neq 0$ is physically necessary, and if such a solution does not exist, the system would have to be concluded as physically inconsistent. Such inconsistency has been shown to exist in the usual non-projectable Hořava-Lifshitz gravity with a generic potential [14], where only the solution $N = 0$ exists when asymptotically flat spaces with appropriate boundary conditions are considered (see [7, 13, 8, 9] for other analyses). When the lapse is fixed to $N = 0$, the Hamiltonian constraints of these theories are generically second-class. The special cases of GR and ultralocal theory are known to be consistent and possess first-class Hamiltonian constraints. Other cases of the potential can at most have first-class Hamiltonian constraints in some parts of space, but not everywhere [26]. Imposing additional constraints may, at least in some cases, provide a way to define a consistent theory. For example, the case of low-energy effective potential of the form R provides a consistent theory when the additional constraint $\pi = 0$ is imposed [14, 15, 16], and even to be equivalent to GR in the gauge $\pi = 0$. Problems related to the modification of the Hamiltonian constraint of GR have also been studied in Ref. [27, 28].

For asymptotically flat spaces we assume the following asymptotic behaviour of the canonical variables in asymptotically flat coordinates ($C = \text{constant}$):

$$\begin{aligned} N &= C + O\left(\frac{1}{r}\right), & N^i &= O\left(\frac{1}{r}\right), \\ g_{ij} &= \delta_{ij} + O\left(\frac{1}{r}\right), & \pi^{ij} &= O\left(\frac{1}{r^2}\right), \\ B &= 1 + O\left(\frac{1}{r}\right), & \pi_B &= O\left(\frac{1}{r^2}\right), \end{aligned} \tag{2.80}$$

which are an extension of the standard boundary conditions of GR [25] that have been used in the usual Hořava-Lifshitz gravity [14]. The auxiliary field A is expected to behave as ${}^4\tilde{R}$ for obvious reasons.

The interpretation of the conditions (2.57) and more generally (2.66) is similar compared to the analysis of the usual Hořava-Lifshitz gravity given in Ref. [14]. They are generically conditions on the lapse N and the only solution that satisfies the appropriate boundary conditions in asymptotically flat space is $N = 0$. Mathematically the constraint $N \approx 0$ is perfectly acceptable, and setting the Lagrange multiplier $\lambda_N = 0$ ensures that this constraint is preserved under dynamics. But as we have stated, this constraint ruins the chance of having any meaningful gravitational dynamics. $N = 0$ implies that the Hamiltonian vanishes in the gauge $N_i = 0$, which means that every function is a constant of motion. Moreover, it is unlikely that the constraint (2.66) could be satisfied by introducing additional constraints that are both consistent under dynamics and do not constrain the canonical variables too much. Such tertiary constraints, together with the existing primary and secondary constraints, would essentially have to imply $E_n \approx 0$ in (2.64)–(2.66). We will demonstrate this by considering the simple special case (2.55).

In the low-energy effective case (2.48) with only one nonvanishing coupling α_1 the condition (2.57) cannot be satisfied by introducing a simple additional constraint like $\pi = 0$ in the usual theory [14, 15, 26, 16]. Instead we could try to introduce more complex constraints to serve the same purpose. In order to satisfy (2.57) the following

three constraints could be imposed:

$$\begin{aligned} \frac{1}{2}\tilde{E}_1^i &= -\frac{2(\alpha_1+1)}{3}\nabla^i\pi + 4(-\alpha_1+\mu)\left(\pi^{ij}-\frac{1}{3}g^{ij}\pi\right)\frac{\nabla_j B}{B} \\ &- \frac{1-2\alpha_1-3\lambda}{3\mu}B\nabla^i\pi_B + \left(\frac{3\lambda+\mu-2\alpha_1-1}{3\mu}+\alpha_1\right)\nabla^i B\pi_B \approx 0. \end{aligned} \quad (2.81)$$

It would require constraints like $\pi(\mathbf{x}) \approx 0$, $\pi_B(\mathbf{x}) \approx 0$ and $B(\mathbf{x}) \approx \text{constant}$ (i.e. “projectable B ”) to make the \tilde{E}_1^i vanish due to relatively simple constraints. But we can see from (2.39)–(2.42) that the preservation of $\pi(\mathbf{x}) \approx 0$ and $\pi_B(\mathbf{x}) \approx 0$ under dynamics would impose additional (quartic) constraints on the canonical variables. These constraints most likely constrain the system too much, leaving too little physical degrees of freedom for gravitational dynamics. If we assume that only two quartic constraints would be needed and that all these five constraints would be second-class and no more constraints are needed to ensure their consistency, and that the variables A and π_A can be eliminated (see Sec. 2.4), we find that the (maximum) number of physical degrees of freedom would be $3/2$ at each point. This is one degree of freedom less than in the usual Hořava-Lifshitz gravity and one half degree of freedom less than in GR. Thus no more than three constraints, e.g. (2.81), can be introduced to satisfy the constraint (2.55) or (2.57). But the consistency of the three extra constraints (2.81) under dynamics is problematic — $\dot{\tilde{E}}_1^i$ is a long and complex expression that appears to imply further (quartic) constraints, which would again lead to the lack of degrees of freedom. In the more general case (2.66) it is practically certain that imposing $E_n \approx 0$ constrains the system too much. A somewhat similar problem has been discovered in the original Hořava-Lifshitz theory [13].

3 Conclusion

As a summary, there are serious problems with the physical consistency of the non-projectable version of our theory, which will likely be impossible to resolve with additional constraints. For the general potential (2.62) the conclusion is similar compared to the usual Hořava-Lifshitz gravity: since $N = 0$ is required by the preservation of the Hamiltonian constraints under time evolution, the theory is physically inconsistent. The difference is that in our $F(R)$ gravity version, the undesirable condition $N = 0$ cannot be avoided consistently even for the low-energy effective action. In the light of Ref. [26], it is very unlikely that there would exist a form of the action with scaling dimension $z = 3$ that could avoid this problem in the usual Hořava-Lifshitz gravity. The same is expected to apply in our theory.

Thus, we can conclude that only the version of the modified $F(R)$ Hořava-Lifshitz gravity with a projectable lapse, $N = N(t)$, is a practicable theory of modified gravity. The non-projectable version of the theory is troubled by a similar physical inconsistency as the usual non-projectable Hořava-Lifshitz gravity.

It would be very interesting to understand whether some further generalization of the Hořava-Lifshitz gravity could be defined consistently without the projectability condition on the lapse — perhaps even along the lines of the general modified first-order Hořava-Lifshitz gravity of Ref. [17]. It would require that the coefficients E_n in (2.64)–(2.66) take a very special form, preferably vanish altogether. Another intriguing prospect would be a generalization of the “healthy extension” of usual Hořava-Lifshitz gravity [10] to our modified $F(R)$ Hořava-Lifshitz gravity.

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